Restrictions on Gauge Groups in Noncommutative Gauge Theory

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We show that the gauge groups $SU(N)$, $SO(N)$ and $Sp(N)$ cannot be realized on a flat noncommutative manifold, while it is possible for $U(N)$.

I. INTRODUCTION

The Yang-Mills theories naturally arise as low energy limits of the theory of open strings. One can obtain Yang-Mills theories with different gauge groups by studying different D-brane configurations (see *e.g.* [1]). For instance, if we place N D-branes on top of each other in the flat space, the corresponding open string theory gives rise to the Yang-Mills theory with the gauge group $U(N)$. One can also obtain gauge theories with other gauge groups such as $SO(N)$ and $Sp(N)$ by using the orientifold construction. In more detail, one combines the spatial reflection $\sigma \to \pi - \sigma$ on the string world-sheet with the target space reflection, $X^\mu \to -X^\mu, \mu = 1, \quad \ldots, k$ and $X^\mu \to X^\mu, \mu = (k+1), \quad \ldots, 10$. It is the goal of this note to study which gauge groups can be realized in the presence of the background B-field when the brane world-volume turns into a noncommutative space [2,3,4].

Interaction with the B-field introduces an extra term into the Polyakov action of the string [4],

$$\Delta S = \frac{-i}{2} \int_\Sigma B_{\mu\nu} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu. \tag{1}$$

Here $a, b = 1, 2$ are world-sheet indices, Σ is the world-sheet and $B_{\mu\nu} = -B_{\nu\mu}$ is the B-field on the target space. One requires the expression (1) to be invariant with respect to the orientifold reflection. This implies the following transformation rules for components of the B-field,

$$\begin{aligned} B_{\|\|} &\to -B_{\|\|} \qquad B_{\perp\perp} \to -B_{\perp\perp} \\ B_{\|\perp} &\to B_{\|\perp} \qquad B_{\perp\|} \to B_{\perp\|}. \end{aligned} \tag{2}$$

Here the symbols $\|$ and $\perp$ stand for the target space indices $\mu = 1, \ldots, k$ and $\mu = (k+1), \ldots, 10$, respectively. In the orientifold construction we finally let the branes lie on the orientifold. The continuity of the B-field implies that the B-field on the brane, $B_{\|\|}$, vanishes. Hence, the brane world-volumes are commutative since it is $B_{\|\|}$ which is responsible for the noncommutativity [2,3,4]. This consideration indicates that one should encounter difficulties in the construction of the gauge theories with gauge groups $SO(N)$ and $Sp(N)$ on noncommutative spaces. Somewhat surprisingly, we find that the reduction from $U(N)$ to $SU(N)$ in the framework of noncommutative geometry also fails.

II. THE CLOSURE OF CLASSICAL LIE ALGEBRAS UNDER THE MOYAL COMMUTATOR

In the flat case the presence of a constant B-field turns the D-branes into noncommutative spaces, with the ordinary pointwise multiplication of functions replaced by the Moyal product,

$$\begin{aligned} (X * Y)(x) &= \exp(\frac{i}{2}\theta^{ij}\partial_i^x \partial_j^y) X(x) Y(y)|_{x=y} = \\ &= XY + \frac{i}{2}\theta^{ij}\partial_i X \partial_j Y + ... \end{aligned} \tag{3}$$

Here X and Y are functions on the D-brane world-volume, and θ^{ij} is a real-valued constant antisymmetric tensor constructed of the metric and B-field [3]. The Moyal product naturally extends to N by N matrices, formula (3) still applies. One can also introduce the Moyal commutator by the formula,

$$[X, Y]_* = X * Y - Y * X. \tag{4}$$

In what follows we check whether the matrix Lie algebras of the classical Lie groups $SO(N), U(N), SU(N)$ and $Sp(N)$ are closed under the Moyal commutator. We choose to work in the fundamental representation of these Lie algebras.

The Lie algebra of $U(N)$ consists of anti-Hermitean matrices, $\overline{X^t} = -X$, where the bar stands for complex conjugation. We first show that this algebra *is* closed under the Moyal commutator. The key observation is the following property of the Moyal product,

$$\overline{(X * Y)^t} = \overline{Y^t} * \overline{X^t}. \tag{5}$$

By using the ordinary rules for the transpose of matrices we get,

$$\begin{aligned} &(X * Y)^t = Y^t X^t + \\ &+ \frac{i}{2}\theta^{ij}\partial_j Y^t \partial_i X^t - \frac{1}{8}\theta^{ij}\theta^{kl}\partial_j\partial_l Y^t \partial_i \partial_k X^t + ... \end{aligned} \tag{6}$$

The construction for higher order terms is obvious. Now we apply the complex conjugation and rename the indices of θ to obtain,

$$\overline{(X*Y)^t} = \overline{Y^t}\,\overline{X^t} + \frac{i}{2}\theta^{ij}\partial_i\overline{Y^t}\partial_j\overline{X^t} - \tag{7}$$

$$\frac{1}{8}\theta^{ij}\theta^{kl}\partial_i\partial_k\overline{Y^t}\partial_j\partial_l\overline{X^t} + ... = \overline{Y^t} * \overline{X^t}. \tag{8}$$

Taking into account $\overline{X^t} = -X$ and $\overline{Y^t} = -Y$ yields,

$$\overline{[X,Y]^t_*} = \overline{(X*Y)^t} - \overline{Y*X)^t} = \tag{9}$$

$$= \overline{Y^t} * \overline{X^t} - \overline{X^t} * \overline{Y^t} = \tag{10}$$

$$= Y*X - X*Y = -[X,Y]_* \tag{11}$$

which shows that the algebra $U(N)$ is closed under the Moyal commutator.

We now turn to the algebras of $SO(N)$, $SU(N)$ and $Sp(N)$. We first show that for $N=2$ these algebras are not closed with respect to the Moyal commutator.

The counter examples for both $SO(2)$ and $Sp(2)$ are given by formulas,

$$X = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}. \tag{12}$$

and the counterexample for $SU(2)$ is

$$X = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} Y = \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix}. \tag{13}$$

Here α and β are coordinates on the manifold chosen so that $\theta^{\alpha\beta} \neq 0$. This can always be done unless $\theta = 0$ and the Moyal product coincides with the ordinary multiplication of matrix-valued functions. With X and Y as given above one can easily compute the Moyal commutator since all derivatives of order higher than one vanish. The result for both counter examples is

$$[X,Y]_* = i\theta^{\alpha\beta}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

Note that this matrix has a nonvanishing trace. Since the Lie algebras of $SO(2), SU(2)$ and $Sp(2)$ consist of traceless matrices, we conclude that they are not closed under the Moyal commutator. This also applies to $SO(N), SU(N)$ and $Sp(N)$ for arbitrary N because they contain $SO(2), SU(2)$ and $Sp(2)$ as subgroups.

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